EXTENSIONS OF D. JACKSON'S THEOREM ON BEST COMPLEX POLYNOMIAL MEAN APPROXIMATION(¹)

BY

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Sufficient conditions for the uniform convergence of polynomials $P_n(z)$ of best qth power approximation to a given function $f(z)$ of a complex variable on a smooth curve Γ were presented in an early paper by Jackson [1]. Although his theorem has found frequent application in approximation theory (see [2]), only slight extensions of the result are to be found in the literature. The present paper gives sufficient conditions for the convergence of the $P_n(z)$ to $f(z)$ in the mean of order $p>q$ on Γ when convergence in the mean on Γ of order p is known for some auxiliary sequence of polynomials, and so includes Jackson's theorem as the special case $p=\infty$. Further extensions of that theorem are given for best approximation by trigonometric polynomials, rational functions with prescribed and free poles, and bounded analytic and meromorphic functions.

The method of proof uses standard best approximation arguments together with a recent result of Walsh [3] which we state as

THEOREM 1. *Let Γ be a Jordan curve of type B and let $P(z)$ be an arbitrary polynomial of degree n (>0). Then for $0<q<p\leqq\infty$ we have for line integral norms on Γ*

$$\|P(z)\|_p \leqq Ln^{1/q-1/p}\|P(z)\|_q, \tag{1}$$

where L is a constant independent of n and z.

A Jordan curve Γ is said to be of *type B* if it is rectifiable, and if there exists a fixed number δ_0 (>0) such that through each point of Γ there passes a circle γ of radius δ_0 whose closed interior lies in the closed interior of Γ.

Our basic result is

THEOREM 2. *Let Γ be a Jordan curve of type B and $f(z)$ a function of class L_p on Γ. If $P_n(z)$ is a sequence of polynomials of respective degrees n of best qth power approximation to $f(z)$ on Γ and $p_n(z)$ is an arbitrary sequence of polynomials of respective degrees n, then for $0<q<p\leqq\infty$,*

$$\|f(z)-P_n(z)\|_p \leqq An^{1/q-1/p}\|f(z)-p_n(z)\|_p. \tag{2}$$

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In particular, a sufficient condition for the $P_n(z)$ to converge to $f(z)$ in the mean of order p on Γ is

$$n^{1/q-1/p}\|f(z)-p_n(z)\|_p \to 0 \quad as\ n \to \infty.$$

Here and below constants A are independent of n and z and may change from one inequality to another.

For $p=\infty$ we interpret the norms in (2) and below as Tchebycheff (uniform) norms on Γ.

We set $S_n(z) \equiv f(z)-p_n(z)$, $\pi_n(z) \equiv p_n(z)-P_n(z)$, whence $f(z)-P_n(z)=S_n(z)+\pi_n(z)$, and set $\varepsilon_n = \|S_n(z)\|_p$, $\mu_n = \|\pi_n(z)\|_p$.

By Theorem 1 we have

$$\mu_n \leqq Ln^{1/q-1/p}\|\pi_n(z)\|_q. \tag{3}$$

Also, by the extremal properties of the $P_n(z)$, we have

$$\begin{aligned} \|\pi_n(z)\|_q &\leqq A_1[\|f(z)-P_n(z)\|_q + \|S_n(z)\|_q] \\ &\leqq 2A_1\|S_n(z)\|_q \leqq A_2\varepsilon_n; \end{aligned} \tag{4}$$

the last inequality follows from the lemma (proved by Hölder's inequality) that, except for a suitable multiplicative constant depending on Γ, the norm is monotonically increasing with respect to increasing order. The constant A_1 above may be chosen equal to 1 if $q \geqq 1$ and equal to $2^{1/q-1}$ if $q<1$.

From (3) and (4) we have the equivalent of (2),

$$\|f(z)-P_n(z)\|_p \leqq A_3(\varepsilon_n+\mu_n) \leqq A_3(\varepsilon_n + A_2Ln^{1/q-1/p}\varepsilon_n) \leqq An^{1/q-1/p}\varepsilon_n.$$

It is worth noting that Theorem 2 holds for arbitrary sequences of polynomials $P_n(z)$, $p_n(z)$ provided merely

$$\|f(z)-P_n(z)\|_q \leqq \|f(z)-p_n(z)\|_q, \quad \text{for } n = 1, 2, \ldots.$$

Also Theorem 1 is valid not merely for a single Jordan curve Γ of type B but for a finite number of mutually exterior such curves, so the extension of Theorem 2 to this configuration follows.

A recent theorem [3, Theorem 11], which is more general in that it applies to arbitrary rather than extremal sequences of polynomials, also yields sufficient conditions for the convergence of the $P_n(z)$. For purposes of comparison we state it as

THEOREM 3. *Let Γ and $f(z)$ be as in Theorem 2 and suppose $q_n(z)$ is a sequence of polynomials of respective degrees n (>0) such that*

$$\|f(z)-q_n(z)\|_q \leqq \sigma_n, \qquad q > 0,\ \sigma_n \to 0,$$

where σ_n is monotonic nonincreasing for n sufficiently large. Then a sufficient condition for the convergence in the mean of order p $(>q)$ of the sequence $q_n(z)$ to $f(z)$ on Γ is

$n^r\sigma_n \to 0$ plus the existence and boundedness of $(2^{m-1} \leqq n < 2^m)$

$$(5)\qquad \begin{aligned} &\frac{2^{mr}\sigma_n + 2^{(m+1)r}\sigma_{2^m} + 2^{(m+2)r}\sigma_{2^{m+1}} + \cdots}{n^r\sigma_n}, \qquad p \geqq 1, \\ &\frac{(2^m)^{pr}\sigma_n^p + (2^{m+1})^{pr}\sigma_{2^m}^p + (2^{m+2})^{pr}\sigma_{2^{m+1}}^p + \cdots}{n^{pr}\sigma_n^p}, \qquad p < 1, \end{aligned}$$

where $r = 1/q - 1/p$. If that condition is satisfied we have

$$\|f(z) - q_n(z)\|_p \leqq An^r\sigma_n.$$

Theorem 3 states sufficient conditions for the convergence of a sequence of polynomials in the mean of order p on Γ when degree of convergence of the sequence is known in a lower order mean. The hypotheses of Theorem 2 imply

$$\|f(z) - P_n(z)\|_q \leqq \|f(z) - p_n(z)\|_q \leqq A\|f(z) - p_n(z)\|_p \equiv A\sigma_n,$$

and hence Theorem 2 is a consequence of Theorem 3 if the σ_n satisfy the conditions of Theorem 3. However, these conditions are not necessarily satisfied for an arbitrary sequence σ_n with $n^r\sigma_n \to 0$. Indeed, take $p = 2$, $0 < q < 2$, $r = 1/q - 1/2$, $\Gamma : |z| = 1$, and

$$(6)\qquad f(z) \equiv \sum_2^\infty a_k z^k, \qquad |a_k|^2 = \frac{2}{k^{2r+1}}\left[\frac{1}{\log^3 k} + \frac{r}{\log^2 k}\right].$$

If $p_n(z)$ is the sum of the first $n-1$ terms of the series in (6), then $p_n(z)$ is the polynomial of degree n of least squares approximation to $f(z)$ on Γ, and

$$\sigma_n^2 = \|f(z) - p_n(z)\|_2^2 = 2\pi \sum_{n+1}^\infty |a_k|^2.$$

Setting

$$g(x) = \frac{2}{x^{2r+1}}\left[\frac{1}{\log^3 x} + \frac{r}{\log^2 x}\right],$$

we have by the integral test

$$1/(n+1)^{2r}\log^2(n+1) = \int_{n+1}^\infty g(x)\,dx \leqq \sum_{n+1}^\infty |a_k|^2 \leqq \int_n^\infty g(x)\,dx = 1/n^{2r}\log^2 n,$$

whence $A_1/n^r \log n \leqq \sigma_n \leqq A_2/n^r \log n$, for some positive constants A_1, A_2. The series in (5) thus diverges like a harmonic series and so Theorem 3 yields no information about the mean square convergence of the polynomials $P_n(z)$ of best qth power approximation to $f(z)$ on Γ. Theorem 2, however, does guarantee convergence in this case.

If the origin lies interior to a Jordan curve Γ, and both Γ and its image under the transformation $w = 1/z$ are of type B, the analogue of Theorem 1 holds [3] for an arbitrary polynomial $P(z, 1/z)$ in z and $1/z$ of degree n (>0). We therefore have by the method of proof of Theorem 2

THEOREM 4. *Let* Γ *have the properties just described and suppose* $f(z)$ *is of class* L_p *on* Γ. *If* $P_n(z, 1/z)$ *is a sequence of polynomials of respective degrees n in z and* $1/z$ *of best qth power approximation to* $f(z)$ *on* Γ, *then for* $0<q<p\leqq\infty$

$$\|f(z)-P_n(z, 1/z)\|_p \leqq An^{1/q-1/p}\|f(z)-p_n(z, 1/z)\|_p,$$

where $p_n(z, 1/z)$ *is an arbitrary comparison sequence of polynomials of respective degrees n in z and* $1/z$.

Theorem 4 has immediate application to approximation by trigonometric polynomials, since any polynomial $p_n(z, 1/z)$ of degree n with $z=e^{i\theta}$ is a trigonometric polynomial in θ of order n. We have the

COROLLARY. *If* $F(\theta)$ *is a function with period* 2π *and* $F(\theta)\in L_p[0, 2\pi]$, *then for* $0<q<p\leqq\infty$ *and norms on* $[0, 2\pi]$

$$\|F(\theta)-t_n(\theta)\|_p \leqq An^{1/q-1/p}\|F(\theta)-s_n(\theta)\|_p,$$

where $t_n(\theta)$ *is a sequence of trigonometric polynomials of respective orders n of best qth power approximation to* $F(\theta)$ *on* $[0, 2\pi]$, *and* $s_n(\theta)$ *is an arbitrary sequence of trigonometric polynomials of respective orders n.*

The analogue of Theorem 2 for approximation by rational functions of degree n is

THEOREM 5. *Let* Γ *and* $f(z)$ *be as in Theorem* 2 *and let K be a closed point set of the extended plane exterior to* Γ. *Call a rational function admissible if its poles lie on K, and for* $n=1, 2, \ldots$ *let* $R_n(z)$ *denote an admissible rational function of degree n of best qth power approximation to* $f(z)$ *on* Γ. *If* $r_n(z)$ *is an arbitrary sequence of admissible rational functions of respective degrees n, then for* $0<q<p\leqq\infty$

$$\|f(z)-R_n(z)\|_p \leqq An^{1/q-1/p}\|f(z)-r_n(z)\|_p.$$

The proof is immediate from a generalization of Theorem 1 which is left to the reader. Compare [4] and [5, p. 231].

Theorem 2 may be further extended to include approximation by sequences of rational functions of types (n, ν) for constant ν, i.e., rational functions of the form

$$\frac{a_0z^n+a_1z^{n-1}+\cdots+a_n}{b_0z^\nu+b_1z^{\nu-1}+\cdots+b_\nu}, \qquad \sum_0^\nu |b_k| \neq 0,$$

provided the finite poles of these rational functions have no limit point on Γ. Such is the case in

THEOREM 6. *Let E be the closed interior of a Jordan curve* Γ *of type B and suppose* $f(z)$ *is meromorphic with precisely* ν *poles in the interior of E and is otherwise finite and continuous on E. If* $R_{n\nu}(z)$ *is a sequence of rational functions of respective types* (n, ν) *of best qth power approximation to* $f(z)$ *on* Γ *and* $r_{n\nu}(z)$ *is an arbitrary sequence of rational functions of respective types* (n, ν), *then for* $0<q<p\leqq\infty$ *we have*

$$\|f(z)-R_{n\nu}(z)\|_p \leqq An^{1/q-1/p}\|f(z)-r_{n\nu}(z)\|_p. \tag{7}$$

The proof requires two lemmas.

LEMMA 1. *Let $f(z)$ and E be as in Theorem 6 and suppose $f_n(z)$ is a sequence of functions each meromorphic interior to Γ with at most ν poles there and otherwise finite and continuous on E. If*

$$\lim_{n\to\infty} \|f(z)-f_n(z)\|_r = 0, \qquad 0 < r \leqq \infty, \tag{8}$$

then for n sufficiently large each $f_n(z)$ has precisely ν poles in the interior of E, which approach respectively the ν poles of $f(z)$ in the interior of E.

Let $q_n(z)=z^{\mu_n}+\cdots+a_n$ denote the polynomial of the form indicated having as its zeros the poles of $f_n(z)$ in the interior of E. By assumption $0\leqq\mu_n\leqq\nu$, and hence the sequence $q_n(z)$ is uniformly bounded on E by the constant max $[(\text{diam } E)^\nu, 1]$. A well-known application of Lagrange's Interpolation Formula thus implies that the $q_n(z)$ form a normal family in the finite plane and each limit function of the family is a polynomial of degree ν.

Let $q(z)$ be such a limit function and $q_{n_i}(z)$ a subsequence which converges uniformly to $q(z)$ on compact sets of the plane. We note that $q(z)$ is monic for there is an integer λ $(0\leqq\lambda\leqq\nu)$ such that infinitely many of the $q_{n_i}(z)$ are of the form $q_{n_i}(z)=z^\lambda+\cdots+a_{n_i}$.

Now let $Q(z)=z^\nu+c_{\nu-1}z^{\nu-1}+\cdots+c_0$ be the polynomial of the form indicated having as its zeros the ν poles of $f(z)$ in the interior of E. We show $q(z)=Q(z)$.

The functions $F(z)\equiv Q(z)q(z)f(z)$, $F_i(z)\equiv Q(z)q_{n_i}(z)f_{n_i}(z)$ are analytic in the interior of E and from (8) and the uniform convergence of the $q_{n_i}(z)$ on Γ satisfy

$$\lim_{i\to\infty} \|F(z)-F_i(z)\|_r = 0.$$

Hence [5, §5.5, Lemma extended] the $F_i(z)$ converge uniformly to $F(z)$ on closed sets in the interior of E, and so the $q_{n_i}(z)f_{n_i}(z)$ converge uniformly to $q(z)f(z)$ on each closed set in the interior of E which contains no pole of $f(z)$. The analyticity of the functions $q_{n_i}(z)f_{n_i}(z)$ thus implies that $q(z)f(z)$ is analytic in the interior of E.

Note, therefore, that if α is a pole of $f(z)$ of order k, then α must be zero of $q(z)$ of order at least k. But since $q(z)$ is of degree ν and $f(z)$ has precisely ν poles in the interior of E, $q(z)$ can have no zeros other than the poles of $f(z)$ and the order of a pole of $f(z)$ must equal its order as a zero of $q(z)$. Since $q(z)$ and $Q(z)$ are monic, it follows that $q(z)=Q(z)$, and since $q(z)$ is an arbitrary limit function of the $q_n(z)$, the $q_n(z)$ converge uniformly to $Q(z)$ on compact sets of the plane. Hurwitz's Theorem then implies that for n sufficiently large the $q_n(z)$ have at least, and hence precisely, ν zeros in the interior of E, which approach respectively the zeros of $Q(z)$.

LEMMA 2. *With $f(z)$ and E as in Theorem 6, let $t_{n\nu}(z)$ denote rational functions of respective types (n, ν) of best rth power approximation to $f(z)$ on Γ. Then*

$$\lim_{n\to\infty} \|f(z)-t_{n\nu}(z)\|_r = 0, \qquad 0 < r \leqq \infty.$$

The function $Q(z)f(z)$ is analytic in the interior of E and continuous on E, and so [5, p. 36] there exists a sequence of polynomials $p_n(z)$ of respective degrees n such that

$$[\max |Q(z)f(z)-p_n(z)|; z \text{ on } \Gamma] \equiv b_n \to 0$$

as $n \to \infty$. Hence,

$$[\max |f(z)-p_n(z)/Q(z)|; z \text{ on } \Gamma] \leqq b_n/m_0,$$

where $m_0 > 0$ is a lower bound for $Q(z)$ on Γ. Since $p_n(z)/Q(z)$ is a rational function of type (n, ν), we have

$$\|f(z)-t_{n\nu}(z)\|_r \leqq \|f(z)-P_n(z)/Q(z)\|_r \leqq A_1 b_n/m_0 \to 0.$$

Proceeding with the proof of Theorem 6, we note that it suffices to prove inequality (7) for the case where the $r_{n\nu}(z)$ are rational functions of type (n, ν) of best pth power approximation to $f(z)$ on Γ. With this assumption we set

$$\pi_{n+\nu,2\nu}(z) \equiv r_{n\nu}(z)-R_{n\nu}(z),$$

and prove the analogue of inequality (1). Clearly $\pi_{n+\nu,2\nu}(z)$ is a rational function of type $(n+\nu, 2\nu)$, and the above lemmas imply that for n sufficiently large the finite poles of the $\pi_{n+\nu,2\nu}(z)$ all lie on a closed set E' interior to Γ. Thus if $d_n(z)=z^{\gamma_n}+\cdots+a_n$ $(0 \leqq \gamma_n \leqq 2\nu)$ is the polynomial of the form indicated having as its zeros the finite poles of $\pi_{n+\nu,2\nu}(z)$, we have for n large enough

$$0 < m_1 \leqq |d_n(z)| \leqq M_1 < \infty, \qquad z \text{ on } \Gamma.$$

Hence,

$$\begin{aligned}\|\pi_{n+\nu,2\nu}(z)\|_p &\leqq \|d_n(z)\pi_{n+\nu,2\nu}(z)\|_p/m_1\\ &\leqq L(n+\nu)^{1/q-1/p}\|d_n(z)\pi_{n+\nu,2\nu}(z)\|_q/m_1\\ &\leqq M_1L_2n^{1/q-1/p}\|\pi_{n+\nu,2\nu}(z)\|_q.\end{aligned}$$

The remainder of the proof of Theorem 6 now follows from the argument of Theorem 2.

It is of some interest to note that Lemmas 1 and 2, whose conclusions are similar to those of J. L. Walsh [7], can be used to extend many theorems on polynomial approximation to approximation by rational functions. These extensions shall be reserved for another occasion.

Further extensions of Theorem 2 for best approximation by bounded analytic and meromorphic functions require

LEMMA 3. *Let Γ be a Jordan curve of type B contained in a simply connected region D of the z-plane and suppose $f_n(z)$ is a sequence of functions analytic in D which satisfy $|f_n(z)| \leqq AR^n$, z in D.*

Then for arbitrary but fixed ρ (>0) and $0<q<p\leqq\infty$ we have for norms on Γ

$$\|f_n(z)\|_p \leqq L_0 n^{1/q-1/p}\|f_n(z)\|_q+O(\rho^n),$$

where L_0 is a constant dependent only on ρ, Γ, and the sequence $f_n(z)$.

From the study of bounded analytic functions [6, §2.2] there exists for each n and N a polynomial $P_{n,N}(z)$ of degree N such that

$$|f_n(z)-P_{n,N}(z)| \leqq AR^n/R_1^N, \qquad z \text{ on } \Gamma,\ R_1 > 1.$$

We choose a positive integer λ so large that $R/R_1^\lambda \equiv \rho_0 < \rho$, whence

$$|f_n(z)-P_{n,\lambda n}(z)| \leqq A\rho_0^n, \qquad z \text{ on } \Gamma.$$

Thus the inequalities

$$\|f_n(z)-P_{n,\lambda n}(z)\|_p \leqq B_1\rho_0^n,$$
$$\|f_n(z)-P_{n,\lambda n}(z)\|_q \leqq B_2\rho_0^n,$$
$$\|P_{n,\lambda n}(z)\|_p \leqq L(\lambda n)^{1/q-1/p}\|P_{n,\lambda n}(z)\|_q \leqq L_1 n^{1/q-1/p}\|P_{n,\lambda n}(z)\|_q$$

imply the following:

$$\begin{aligned}\|f_n(z)\|_p &\leqq B_3[\|P_{n,\lambda n}(z)\|_p+\rho_0^n]\\ &\leqq B_3[L_1 n^{1/q-1/p}\|P_{n,\lambda n}(z)\|_q+\rho_0^n]\\ &\leqq B_3[L_1 n^{1/q-1/p}(B_4\|f_n(z)\|_q+B_5\rho_0^n)+\rho_0^n]\\ &\leqq L_0 n^{1/q-1/p}\|f_n(z)\|_q+O(\rho^n).\end{aligned}$$

We have established the result needed for the proof of

THEOREM 7. *Let Γ and D be as in Lemma* 3 *and suppose $f(z)$ is of class L_p on Γ. Let a sequence M_n ($\geqq 0$) ($n=1, 2, \ldots$) be given which satisfies $M_n \leqq B_0 R^n$ for some constants $R>1$, $B_0>0$, and for each $n=1, 2, \ldots$, let $f_n(z)$ be a function analytic in D which among all functions $h(z)$ analytic in D with $|h(z)| \leqq M_n$ for z in D, minimizes $\|f(z)-h(z)\|_q$. Then a sufficient condition for the $f_n(z)$ to converge to $f(z)$ in the mean of order p, $0<q<p\leqq\infty$, on Γ is*

$$n^{1/q-1/p}\|f(z)-g_n(z)\|_p \to 0 \quad \textit{as } n\to\infty,$$

for some sequence of functions $g_n(z)$ analytic in D with $|g_n(z)| \leqq M_n$ for z in D.

Moreover, for arbitrary but fixed ρ ($0<\rho<1$)

$$\|f(z)-f_n(z)\|_p \leqq An^{1/q-1/p}\|f(z)-g_n(z)\|_p+O(\rho^n).$$

To study approximation by meromorphic functions of bounded type we first give the

DEFINITION. A meromorphic function $F(z)$ is said to be of *type* (M, ν) *in a domain D* if $F(z)$ is of the form $F(z)=h(z)/q(z)$, where $h(z)$ is analytic in D with $|h(z)| \leqq M$ there and $q(z)=z^\mu+a_{\mu-1}z^{\mu-1}+\cdots+a_0$ ($0\leqq\mu\leqq\nu$).

We suppose a function $f(z)$ is defined and continuous on a rectifiable Jordan curve Γ contained in a bounded domain D. It follows that for each $M\geqq 0$ and integral $\nu\geqq 0$ there exists a meromorphic function of type (M, ν) in D of best qth power approximation to $f(z)$ on Γ. We briefly indicate this fact. Let

$$b = \inf \|f(z)-F(z)\|_q, \qquad 0 < q \leqq \infty,$$

taken over all functions $F(z)$ of type (M, ν) in D and let $F_k(z)$ be a sequence of functions of type (M, ν) in D which satisfy

$$\|f(z)-F_k(z)\|_q \to b \quad \text{as } k \to \infty.$$

Write $F_k(z)=h_k(z)/q_k(z)$ as in the above definition. For fixed k let $\alpha_1, \ldots, \alpha_j$ be the zeros of $q_k(z)$ which lie at a distance greater than one from the closure of D and let $\beta_1, \ldots, \beta_i$ $(i \leqq \nu)$ be the remaining zeros of $q_k(z)$. Then $F_k(z)$ has the representation

$$F_k(z) = H_k(z)/Q_k(z),$$

where $H_k(z)=h_k(z)/(z-\alpha_1)\cdots(z-\alpha_j)$, $Q_k(z)=(z-\beta_1)\cdots(z-\beta_i)$ and $|H_k(z)| \leqq M$ for z in D. We have thus shown that it suffices to assume that the zeros of the $q_k(z)$ are uniformly bounded.

With this assumption, the $h_k(z)$ and the $q_k(z)$ are uniformly bounded in D, and hence it is possible to extract a subsequence $\Phi_j(z)$ from the $F_k(z)$ which converges uniformly on closed sets of a domain D' obtained from D by the omission of at most ν points. If $F_0(z)$ denotes the limit function of the $\Phi_j(z)$, it is easy to see that $F_0(z)$ is also of type (M, ν) in D and the reasoning of [5, §12.5] implies

$$b = \|f(z)-F_0(z)\|_q.$$

THEOREM 8. *Let E, Γ and $f(z)$ be as in Theorem 6 and let D be a bounded simply connected domain containing Γ in its interior. For each $n=1, 2, \ldots$, let $F_n(z)$ be a meromorphic function of type (M_n, ν) in D of best qth power approximation to $f(z)$ on Γ, where $M_n \to \infty$ and $M_n \leqq B_0R^n$ for some constants $R>1$, $B_0>0$. Then a sufficient condition for the $F_n(z)$ to converge to $f(z)$ in the mean of order p, $0<q<p\leqq\infty$, on Γ is*

$$n^{1/q-1/p}\|f(z)-G_n(z)\|_p \to 0, \qquad n \to \infty,$$

for some sequence $G_n(z)$ of meromorphic functions of respective types (M_n, ν) in D.

Moreover, for arbitrary but fixed ρ $(0<\rho<1)$

$$\|f(z)-F_n(z)\|_p \leqq An^{1/q-1/p}\|f(z)-G_n(z)\|_p+O(\rho^n).$$

With Lemma 1 and Lemma 3 at our disposal the proof merely requires an analogue of Lemma 2. Let $T_n(z)$ denote a sequence of meromorphic functions of respective types (M_n, ν) in D of best rth power approximation to $f(z)$ on Γ. We need to show

$$\lim_{n\to\infty} \|f(z)-T_n(z)\|_r = 0, \qquad 0 < r \leqq \infty.$$

Let $m_n=[\sup |p_n(z)|; z \text{ in } D]$, where the $p_n(z)$ are the polynomials in the proof of Lemma 2. We have shown that for each $\varepsilon>0$ there exists an integer i such that

$$\|f(z)-p_i(z)/Q(z)\|_r < \varepsilon.$$

Now choose an integer N so large that $M_n \geqq m_t$ for $n \geqq N$. The extremal property of the $T_n(z)$ then implies

$$\|f(z)-T_n(z)\|_r < \varepsilon \quad \text{for } n \geqq N.$$

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